

# Euclidean supersymmetry, twisting and topological sigma models

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**ABSTRACT:** We discuss two dimensional  $N$ -extended supersymmetry in Euclidean signature and its R-symmetry. For  $N = 2$ , the R-symmetry is  $SO(2) \times SO(1,1)$ , so that only an  $A$ -twist is possible. To formulate a  $B$ -twist, or to construct Euclidean  $N = 2$  models with  $H$ -flux so that the target geometry is generalised Kahler, it is necessary to work with a complexification of the sigma models. These issues are related to the obstructions to the existence of non-trivial twisted chiral superfields in Euclidean superspace.

**KEYWORDS:** Extended Supersymmetry, Superspaces, Topological Strings, Sigma Models.

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## 1. Introduction

The construction of the topological sigma model by twisting the  $(2, 2)$  supersymmetric sigma model pioneered in [1] and further discussed in, e.g., [2, 3], explicitly or implicitly assumes the existence of an underlying  $(2, 2)$  Euclidean supersymmetry. In this letter we analyse such supersymmetries and show that, strictly speaking, the  $R$  symmetry group does not allow for both an  $A$  and a  $B$  twist, but only an  $A$  twist. In 2D Lorentzian space,  $(2, 2)$  supersymmetry has R-symmetry  $SO(2) \times SO(2)$ . One might expect that, after Wick rotating so that the Lorentz group becomes  $SO(2)$ , the resulting theory should have  $SO(2) \times SO(2) \times SO(2)$  symmetry, allowing one to twist the  $SO(2)$  Lorentz symmetry with the diagonal subgroup of the  $SO(2) \times SO(2)$  R-symmetry to give the  $A$ -twist or the anti-diagonal subgroup of the  $SO(2) \times SO(2)$  R-symmetry to give the  $B$ -twist. However, this Wick rotation with  $SO(2) \times SO(2) \times SO(2)$  symmetry gives a theory which is not supersymmetric, so that the twisted versions would not automatically have the desired BRST symmetry. Here we analyse  $(2, 2)$  supersymmetry in Euclidean 2D space, and find that the R-symmetry is not  $SO(2) \times SO(2)$  but is instead  $SO(2, \mathbb{C}) = SO(2) \times SO(1, 1)$ . This allows an  $A$ -twist with the  $SO(2)$  subgroup of the R-symmetry, but not a  $B$ -twist. The  $B$ -twist requires going to the complexification of the theory. Indeed, this is implicit already in the early work on the subject. In 2D Euclidean space, left-handed and right-handed fermions are related by complex conjugation. The B-model has different twists for left and right-handed fermions, requiring them to be treated as independent so that one is formally dealing with the complexified model.

These issues can be, and usually are, suppressed in the discussion of topological sigma-models with Calabi-Yau target spaces. However, they become important in discussing topological sigma-models with H-flux, so that the target space has Generalized Kähler Geometry. As in the analysis of the models in [4], one needs to consider a complexified version of the Euclidean sigma model. One place where a careful treatment of these issues is particularly relevant is in understanding whether or not the Wess-Zumino term has the factor of ‘ $i$ ’ one would expect for a Euclidean sigma-model. Indeed, it was seen in [4]

that some parts of the Wess-Zumino term in the twisted model are imaginary and have an interpretation in terms of gerbes, while others are real and contribute to a complexified Kahler class. Another place where one can see that there is a problem is in the (2,2) super-space formulation of the sigma-models. In Lorentzian signature, the general (2,2) sigma model can be written in terms of chiral, twisted chiral and semi-chiral superfields [10, 7]. In Euclidean signature, as we will review below, twisted chiral superfields are problematic and there is no sensible way of continuing twisted chiral superfields to Euclidean signature, unless one goes to the complexified model.

## 2. Supersymmetry algebra

The  $(p, q)$  Lorentzian (pseudo) supersymmetry algebra in  $2D$  is given by<sup>1</sup>[5, 6]

$$\begin{aligned} \{Q_+^I, Q_+^J\} &= 2i\eta^{IJ}\partial_+, & I, J &= 1, \dots, p, \\ \{Q_-^{I'}, Q_-^{J'}\} &= 2i\eta^{I'J'}\partial_-, & I', J' &= 1, \dots, q, \end{aligned} \tag{2.1}$$

where the supercharges  $Q_\pm = Q_\pm^\dagger$  are Majorana-Weyl spinors of chirality  $\pm 1$ . Ordinary supersymmetry corresponds to  $\eta^{IJ} = \delta^{IJ}$ ,  $\eta^{I'J'} = \delta^{I'J'}$  while for pseudo supersymmetry  $\eta^{IJ}, \eta^{I'J'}$  are arbitrary symmetric matrices, which we shall take to be invertible. (We shall not discuss the possibility of central charges here.) The group of automorphisms of the algebra (2.1) include transformations

$$\begin{aligned} Q_+^I &\rightarrow M^I{}_J Q_+^J & : M^t \eta M &= \eta, \\ Q_-^{I'} &\rightarrow \tilde{M}^{I'}{}_{J'} Q_-^{J'} & : \tilde{M}^t \eta' \tilde{M} &= \eta'. \end{aligned} \tag{2.2}$$

To preserve the Majorana-Weyl conditions, the matrices  $M, \tilde{M}$  are real and independent. Thus, the group of automorphisms is (space-time  $\times$  R-symmetry)

$$\text{SO}(1, 1) \times \text{SO}(n, p - n) \times \text{SO}(m, q - m), \tag{2.3}$$

where  $n(m)$  denotes the number of positive eigenvalues of  $\eta$  ( $\eta'$ ). For ordinary  $(p, q)$  supersymmetry with  $\eta = \delta$  and  $\eta' = \delta$ , the group is

$$\text{SO}(1, 1) \times \text{SO}(p) \times \text{SO}(q). \tag{2.4}$$

In Euclidean signature there are no Majorana-Weyl fermions but we may use complex Weyl fermions. Hermitian conjugation changes the chirality according to

$$(Q_\pm)^\dagger = Q_\mp. \tag{2.5}$$

This means that we must have an equal number of left and right supersymmetries,  $p = q := N$ . Since the charges are now complex, the R-symmetry transformations can be generalised to allow the matrices  $M$  in (2.2) to be complex. Then the R-symmetry transformations are

$$Q_+^I \rightarrow M^I{}_J Q_+^J \quad : M^t \eta M = \eta. \tag{2.6}$$

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<sup>1</sup>We have changed nomenclature from the original “twisted-” to “pseudo-” supersymmetry to avoid confusion when discussing another twist below.

This implies that the complex matrices  $M \in \text{SO}(N, \mathbb{C})$ , so that in Euclidean signature, the group of automorphisms of  $N$ -extended supersymmetry is

$$\text{SO}(2) \times \text{SO}(N, \mathbb{C}) . \tag{2.7}$$

Note that the negative chirality supercharges transform under the complex conjugate transformations

$$Q_-^I \rightarrow \bar{M}^I{}_J Q_-^J . \tag{2.8}$$

### 3. Twisting and sigma models

Twisting an  $N = 2$  supersymmetric Euclidean theory in  $2D$  involves selecting an  $\text{SO}(2)$  subgroup of the  $R$  symmetry group and then twisting the  $2D$  Lorentz group  $\text{SO}(2)$  with the  $\text{SO}(2)$  R-symmetry subgroup, so that the new Lorentz group is an  $\text{SO}(2)$  subgroup of this  $\text{SO}(2) \times \text{SO}(2)$ . We see from the above that the  $R$  symmetry group for the  $(2, 2)$  model is

$$\text{SO}(2, \mathbb{C}) = \text{SO}(2) \times \text{SO}(1, 1) . \tag{3.1}$$

There is then a unique choice of  $\text{SO}(2)$  subgroup of the R-symmetry group, and twisting with this gives an  $A$ -twist. A  $B$ -twist is not possible for realisations of this supersymmetry, as in going to the Euclidean theory, the second  $\text{SO}(2)$  of the Lorentzian R-symmetry has become an  $\text{SO}(1, 1)$ .

We now turn to the application of our discussion to  $(2, 2)$  supersymmetric sigma models. A useful starting point is the  $N = 1$  supersymmetric sigma-model in 4D Lorentzian spacetime. This has a Kahler target space and  $\text{SO}(2)$  R-symmetry [11]. It can be formulated in terms of chiral superfields  $\phi$ , with  $N = 1$  superspace lagrangian given by the Kahler potential  $K(\phi, \bar{\phi})$ . Dimensionally reducing from  $3 + 1$  dimensions on two spacelike dimensions gives a theory in  $1 + 1$  dimensions with R-symmetry  $\text{SO}(2) \times \text{SO}(2)$ , with the extra  $\text{SO}(2)$  arising from rotation symmetry in the two internal dimensions. Alternatively, reducing on one space and one time dimension gives a theory in two Euclidean dimensions with R-symmetry  $\text{SO}(2) \times \text{SO}(1, 1)$ , with the extra  $\text{SO}(1, 1)$  arising from Lorentz transformations in the two internal dimensions. In both cases, dimensional reduction ensures  $N = 2$  supersymmetry in the reduced theory, and the reduction gives a natural understanding of the R-symmetry groups in the two cases. In both cases, the theory can be written in  $N = 2$  superspace in terms of chiral superfields  $\phi$  and their complex conjugates, anti-chiral superfields  $\bar{\phi}$  satisfying the constraints

$$\bar{\mathbb{D}}_{\pm} \phi = 0, \quad \mathbb{D}_{\pm} \bar{\phi} = 0, \tag{3.2}$$

with the standard supercovariant derivatives

$$\{\mathbb{D}_{\pm}, \bar{\mathbb{D}}_{\pm}\} = 2i\partial_{\pm} . \tag{3.3}$$

In Euclidean signature  $\partial_{++} \rightarrow \partial, \partial_{--} \rightarrow \bar{\partial}$ . The chirality constraints (3.2) make sense in Euclidean signature as well as Lorentzian.

Consider now the extension of these models to include a Wess-Zumino term. For the  $N = 2$  sigma model on a Lorentzian 2D world-sheet, the target space is then a bihermitian geometry [9], recently recast as a generalised Kahler geometry [12]. The off-shell models of [9] are formulated in  $N = 2$  superspace with both chiral superfields  $\phi$  and twisted chiral superfields  $\chi$ , which satisfy the Lorentzian constraints

$$\bar{\mathbb{D}}_+\chi = \mathbb{D}_-\chi = 0, \quad \mathbb{D}_+\bar{\chi} = \bar{\mathbb{D}}_-\bar{\chi} = 0. \quad (3.4)$$

The superspace lagrangian is then a generalised Kahler potential  $K(\phi, \bar{\phi}, \chi, \bar{\chi})$ . A complex coordiante transformation in superspace exchanges the constraints (3.2) and (3.4). A model in 1+1 dimensions with *only* twisted chiral fields is thus equal to one with *only* chiral fields. The general case has semi-chiral superfields as well as chiral and twisted chiral superfields [10].

The natural expectation would be that the version of these models with Euclidean world-sheet should again have chiral and twisted chiral superfields. However, there is a problem with twisted chiral superfields in superspace, as was first realised in [7]. In Euclidean signature, the conjugation relations  $(\mathbb{D}_\pm)^\dagger = \bar{\mathbb{D}}_\mp$  imply that conjugating the constraints (3.4) give

$$\bar{\mathbb{D}}_-\chi = \mathbb{D}_+\chi = 0, \quad \mathbb{D}_-\bar{\chi} = \bar{\mathbb{D}}_+\bar{\chi} = 0, \quad (3.5)$$

which together with (3.4) force  $\chi, \bar{\chi}$  to be constant. If instead one takes the constraints  $\bar{\mathbb{D}}_+\chi = \mathbb{D}_-\chi = 0$  plus their conjugates  $\mathbb{D}_-\bar{\chi} = \bar{\mathbb{D}}_+\bar{\chi} = 0$ , then only the  $\chi$ -independent part of the potential  $K(\phi, \bar{\phi}, \chi, \bar{\chi})$  contributes to the geometry, and this reduces to the usual Kahler case in terms of chiral superfields only. There is one final possibility that does not involve complexifying the twisted chiral fields. That is to have a real superfield  $\chi$  satisfying the twisted chiral constriant  $\bar{\mathbb{D}}_+\chi = \mathbb{D}_-\chi = 0$  and an independent real twisted anti chiral superfield  $\tilde{\chi}$  satisfying  $\mathbb{D}_+\tilde{\chi} = \bar{\mathbb{D}}_-\tilde{\chi} = 0$ . The superspace lagrangian  $K(\phi, \bar{\phi}, \chi, \tilde{\chi})$  then gives an interesting (2,2) sigma-model in Euclidean space, but with a target space of indefinite signature which is not generalised Kahler; this model will be discussed elsewhere [8].

So far we have limited the discussion to (2,2) sigma models described by chiral and twisted chiral fields. To describe a general (2,2) model, semi-chiral superfields are also needed [7, 10]. In Lorentzian signature the left and right semi-(anti)chiral superfields obey the constraints

$$\begin{aligned} \bar{\mathbb{D}}_+\mathbb{X}_L &= 0, & \bar{\mathbb{D}}_-\mathbb{X}_R &= 0, \\ \mathbb{D}_+\bar{\mathbb{X}}_L &= 0, & \mathbb{D}_-\bar{\mathbb{X}}_R &= 0. \end{aligned} \quad (3.6)$$

and there is a local formulation in terms of chiral, twisted chiral and semi-chiral superfields, with a generalised Kahler potential  $K(\phi, \bar{\phi}, \chi, \bar{\chi}, \mathbb{X}_L, \bar{\mathbb{X}}_L, \mathbb{X}_R, \bar{\mathbb{X}}_R)$ . A potential depending on only one kind of semi-chiral superfield,  $K(\phi, \bar{\phi}, \chi, \bar{\chi}, \mathbb{X}_L, \bar{\mathbb{X}}_L)$  say, does not have a standard kinetic term for the components of  $\mathbb{X}_L$ , so that the model has a topological nature in this sector [7].

For world-sheets of Euclidean signature, one can similarly introduce semi-chiral fields  $\mathbb{Y}_L, \mathbb{Y}_R$ , but now the constraints consistent with complex conjugation are

$$\bar{\mathbb{D}}_+\mathbb{Y}_L = 0, \quad \bar{\mathbb{D}}_-\mathbb{Y}_R = 0, \quad \mathbb{D}_+\bar{\mathbb{Y}}_R = 0, \quad \mathbb{D}_-\bar{\mathbb{Y}}_L = 0. \quad (3.7)$$

Now a generalised Kahler potential  $K(\phi, \bar{\phi}, \chi, \bar{\chi}, \mathbb{Y}_L, \bar{\mathbb{Y}}_L, \mathbb{Y}_R, \bar{\mathbb{Y}}_R)$  gives a kinetic term for the components of the semi-chiral superfields which is non-positive, constructed from a metric of indefinite signature. The change in the constraints means that a potential depending on only one kind of semi-chiral superfield,  $K(\phi, \bar{\phi}, \chi, \bar{\chi}, \mathbb{Y}_L, \bar{\mathbb{Y}}_L)$  gives a standard kinetic term for the components of  $\mathbb{Y}_L$ . The geometry of these models containing semi-chiral fields will be discussed elsewhere.

#### 4. Complexification

In order to formulate a Euclidean version of the supersymmetric sigma models with generalised Kahler targets, or to formulate a  $B$ -twist, it is necessary to work with complexified theories in which positive and negative chirality fields are treated as independent and are no longer complex conjugate, as they would be in Euclidean space. The complex worldsheet coordinates  $z, \bar{z}$  are treated as independent complex variables rather than as complex conjugates (as often done in conformal field theory), and the metric

$$ds^2 = 2dzd\bar{z} \tag{4.1}$$

is preserved by the complexified Lorentz group

$$\text{SO}(2, \mathbb{C}) \simeq \mathbb{C}^* \simeq \text{SO}(2) \times \text{SO}(1, 1) \tag{4.2}$$

under which  $z \rightarrow az, \bar{z} \rightarrow a^{-1}\bar{z}$  for  $a \in \mathbb{C}^*$ . The positive chirality supercharges  $Q_+$  are regarded as independent of the negative chirality ones  $Q_-$ , so that again we can have  $(p, q)$  supersymmetry with algebra (2.1). The automorphisms are again of the form (2.2) but with  $M^I{}_J$  and  $\tilde{M}^I{}_J$  independent complex matrices, so that the R-symmetry group is  $\text{SO}(p, \mathbb{C}) \times \text{SO}(q, \mathbb{C})$ , and the full symmetry group is

$$\text{SO}(2, \mathbb{C}) \times \text{SO}(p, \mathbb{C}) \times \text{SO}(q, \mathbb{C}) . \tag{4.3}$$

In particular, for (2,2) supersymmetry, this group becomes

$$\text{SO}(2, \mathbb{C}) \times \text{SO}(2, \mathbb{C}) \times \text{SO}(2, \mathbb{C}) \tag{4.4}$$

and allows both an  $A$ -twist and a  $B$ -twist, as well as a half-twist.

In superspace, one can introduce chiral superfields  $\phi$  and independent anti-chiral superfields  $\bar{\phi}$  satisfying the constraints (3.2) together with twisted chiral superfields  $\chi$  and independent twisted anti-chiral superfields  $\bar{\chi}$  satisfying the constraints (3.4) and the superspace lagrangian is again a generalised Kahler potential  $K(\phi, \bar{\phi}, \chi, \bar{\chi})$ . This is consistent so long as  $\phi, \bar{\phi}, \chi, \bar{\chi}$  are treated as independent complex fields, and gives a target geometry which is a complexification of generalised Kahler geometry. This allows both an  $A$ -twist and a  $B$ -twist, and it was the twisting of this complexified sigma-model that was analysed in [4].

Similarly, one can introduce left and right semi-chiral superfields  $\mathbb{X}_L, \mathbb{X}_R$  and independent anti-semi-chiral ones  $\bar{\mathbb{X}}_L, \bar{\mathbb{X}}_R$  satisfying the constraints (3.6). Then the general superspace lagrangian is given by a generalised Kahler potential  $K(\phi, \bar{\phi}, \chi, \bar{\chi}, \mathbb{X}_L, \bar{\mathbb{X}}_L, \mathbb{X}_R, \bar{\mathbb{X}}_R)$ .

This complexified geometry gives a Euclideanisation of the standard Lorentzian signature sigma model with generalized Kähler target geometry. From the superspace point of view, when all fields are complexified we can have chiral, twisted chiral and semi-chiral superfields in the model. This is necessary, e.g., to be able to discuss twisting, mirror symmetry or T-duality in superspace. We plan to return to Euclidean  $(2, 2)$  sigma models in superspace in a separate publication [8]

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